Commun. Math. Phys. 174, 43 – 56 (1995)

Communications in
Mathematical
Physics

Complexity of Trajectories in Rectangular Billiards

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Received: 20 May 1994/in revised form: 30 January 1995

Abstract: To a trajectory of the billiard in a cube we assign its symbolic trajectory – the sequence of numbers of coordinate planes, to which the faces met by the trajectory are parallel. The complexity of the trajectory is the number of different words of length n occurring in it. We prove that for generic trajectories the complexity is well defined and calculate it, confirming the conjecture of Arnoux, Mauduit, Shiokawa and Tamura [AMST].

0. Introduction

Consider a rectangular billiard in $\mathbb{R}^{s+1}$, that is the dynamical system defined by the free motion of the point between collisions with the boundary of the billiard domain and elastic reflections at the collision instants, with the billiard domain being a $(s+1)$-dimensional cube with the faces parallel to coordinate planes.

This dynamical system is equivalent to the trivial system with constant velocities on a torus and is studied in much detail (see [T] for a survey). However, there are questions still attracting a lot of attention in the literature, such as the question of the coding of trajectories by listing its consecutive collisions with the boundary.

Specifically, to a trajectory one associates an infinite word in alphabet $\mathscr{A} = \{\mathbf{0}, \dots, \mathbf{s}\}$ as follows: each time the trajectory meets a face of the cube parallel to the j^{th} coordinate plane, one writes down $\mathbf{j}$. The resulting infinite word will be called a *symbolic trajectory*. In exceptional cases the trajectory meets more than one face simultaneously, but such cases are not generic and will not be considered.

The resulting symbolic trajectories arise in numerous problems related to number theory, quasicrystals, computer graphics, etc. These trajectories were abundantly studied in the two-dimensional case (where they bear also such names as Sturmian trajectories or Beatty or Wythoff sequences); a sample bibliography can be found in [B, LP, S]. Although multidimensional generalizations are investigated much less, one can find quite a lot of results on those in the papers mentioned.

★ The author was supported by DFG.