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Asymptotics for the Fredholm Determinant of the Sine Kernel on a Union of Intervals

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Abstract: In the bulk scaling limit of the Gaussian Unitary Ensemble of hermitian matrices the probability that an interval of length s contains no eigenvalues is the Fredholm determinant of the sine kernel $\frac{\sin(x-y)}{\pi(x-y)}$ over this interval. A formal asymptotic expansion for the determinant as s tends to infinity was obtained by Dyson. In this paper we replace a single interval of length s by sJ, where J is a union of m intervals and present a proof of the asymptotics up to second order. The logarithmic derivative with respect to s of the determinant equals a constant (expressible in terms of hyperelliptic integrals) times s, plus a bounded oscillatory function of s (zero if $m = 1$, periodic if $m = 2$, and in general expressible in terms of the solution of a Jacobi inversion problem), plus $o(1)$. Also determined are the asymptotics of the trace of the resolvent operator, which is the ratio in the same model of the probability that the set contains exactly one eigenvalue to the probability that it contains none. The proofs use ideas from orthogonal polynomial theory.

I. Introduction

The sine kernel

$$K(x, y) := \frac{\sin(x - y)}{\pi(x - y)}$$

arises in many areas of mathematics and mathematical physics. There is an extensive literature on the asymptotics of the eigenvalues of K_s, the operator with this kernel on an interval of length s, as $s \to \infty$, for example [4, 6, 12, 15], and asymptotic formulas of various kinds have been obtained. Some of these derivations were rigorous, others were more heuristic. The Fredholm determinant of this kernel, i.e., the operator determinant $\det(I - K_s)$, is of particular interest. In the bulk scaling limit of the Gaussian Unitary Ensemble of hermitian matrices it equals the probability that an interval of length s contains no eigenvalues.

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