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The Spectral Theory of the Vibrating Periodic Beam

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Abstract: We study the spectral theory of the fourth-order eigenvalue problem

$$[a(x)u''(x)]'' = \lambda\rho(x)u(x), \quad -\infty < x < \infty ,$$

where the functions a and ρ are periodic and strictly positive. This equation models the transverse vibrations of a thin straight (periodic) beam whose physical characteristics are described by a and ρ.

We examine the structure of the spectrum establishing the fact that the periodic and antiperiodic eigenvalues are the endpoints of the spectral bands. We also introduce an entire function, which we denote by $E(\lambda)$, connected to the spectral theory, whose zeros (at least the ones of odd multiplicity) are shown to lie on the negative real axis, where they define a collection of "pseudogaps." Next we prove some inverse results in the spirit of two old theorems of Borg for the Hill's equation. We finish with a "determinant formula" (i.e. a multiplicative trace formula) and some comments on its role in the formulation of the general inverse problem.

1. Introduction

The *Euler–Bernoulli equation for the free undamped infinitesimal transverse vibrations of a thin, straight beam* can be written as (see [T-Y] or [G])

$$[a(x)u''(x)]'' = \lambda\rho(x)u(x) ,$$

where u is the deflection of the beam and the *positive* functions a and ρ correspond to physical characteristics of the beam.

Mathematically speaking, the Euler–Bernoulli equation (together with appropriate boundary conditions) is a fourth-order eigenvalue problem. In contrast with the second-order case, works on fourth-order problems appear sparsely in the literature. Joyce McLaughlin (see [Mc] or [G]; also [B] for some relevant results) has solved an inverse problem, where the beam equation is considered on a finite interval with certain separated boundary conditions. A variant of this problem was recently solved by V.A. Yurko (see [Y1]).