Commun. Math. Phys. 170, 249 – 281 (1995)

Communications in Mathematical Physics

Dynamical Approximation Entropies and Topological Entropy in Operator Algebras

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Received: 25 April 1994 / in revised form 24 August 1994

Abstract: Dynamical entropy invariants, based on a general approximation approach are introduced for C*- and W*-algebra automorphisms. This includes a noncommutative extension of topological entropy.

The Connes–Störmer entropy [4] and its generalization by Connes–Narnhofer–Thirring [3] (see [9] for a recent alternative approach) extend the entropy invariant of Kolmogorov to the context of W^*-algebra automorphisms. These entropies may be viewed as "observable"-entropies, i.e. they are based on the physics point of view of observing the quantum dynamical system via abelian models. Here we explore another route to entropy based on approximation (we also briefly discussed the approximation idea in [14]). One may think of approximation entropies as "growth"-entropies, reflecting the mathematical idea of the growth of the algebra produced by the automorphism. The invariants we obtain are $\geqq$ the usual ones and we show equality for non-commutative Bernoulli shifts and in the commutative cases. Note also that "observable" entropy is essential in getting lower bounds for approximation entropies, while approximation entropies give useful upper bounds for "observable" entropy. Let us also mention from the beginning that the natural framework for Connes–Störmer. Connes–Narnhofer–Thirring and approximation entropies is that of algebras satisfying hyperfiniteness or nuclearity assumptions.

There are several reasons for studying approximation entropies. One motivation is the search for a non-commutative analogue to McMillan's theorem for the Connes–Störmer entropy. Proving equality of the Connes-Störmer entropy and of the approximation entropy for a given automorphism can be viewed as a kind of weak McMillan-type of theorem.

For approximation entropies, the entropy of a tensor product of two automorphisms is bounded by the sum of the entropies of the automorphisms. Powers' shifts, which are very far from asymptotically abelian have been shown recently by Narnhofer–Störmer–Thirring [15] to provide a counterexample to this tensor product property for the Connes–Störmer entropy. For these Powers' shifts the Connes–

* Supported in part by Grant No. DMS 9204174 from the National Science Foundation.