Commun. Math. Phys. 170, 207–217 (1995)

Communications in Mathematical Physics

An Unconstrained Hamiltonian Formulation for Incompressible Fluid Flow

John H. Maddocks, Robert L. Pego
Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA

Received April 19, 1994

Abstract: The equations governing the time evolution of an ideal fluid in material coordinates are expressed as an unconstrained canonical Hamiltonian system. The incompressibility of the flow is consequent upon certain first integrals of the motion. The variable conjugate to the configuration field is not the usual linear momentum, but is instead a quantity that is related to linear momentum through an auxiliary scalar field whose time derivative is the pressure. The definition of the Hamiltonian involves a minimization with respect to this auxiliary field. The method of derivation may be generally applied to obtain unconstrained Hamiltonian descriptions of Lagrangian field equations subject to pointwise constraints.

1. Introduction

In this note we present a novel Hamiltonian formulation of the equations of ideal fluid flow, expressed in material coordinates. The Hamiltonian system is described by Eqs. (6) below, which have the following noteworthy features: First, the variables are *unconstrained* – incompressibility is not guaranteed from configuration constraints, but rather arises from integrals of the evolution. Offsetting this, the definition of the Hamiltonian involves a *minimization* with respect to an auxiliary field. An associated feature is that the conjugate momentum has an *unobservable* component that does not affect the physical flow.

Surveys and extensive references describing prior variational and Hamiltonian formulations of incompressible, inviscid fluid flow are given by Benjamin [2], Holm et al. [9], and Serrin [13], for example. Our description arises from a general approach for constrained systems that is not restricted to problems in fluid mechanics. Rather, it is widely applicable for obtaining unconstrained Hamiltonian dynamical systems from Lagrangian field equations that are subject to pointwise constraints. We first came upon the approach in the context of the dynamics of an inextensible elastic rod [7], where the associated Hamiltonian system facilitated the characterization

---

Correspondence to: R. L. Pego