Commun. Math. Phys. 166, 317–336 (1994)

Communications in Mathematical Physics

A Four-Thirds law for phase randomization of stochastically perturbed oscillators and related phenomena

Robert Cogburn, James A. Ellison★

Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, New Mexico 87131, USA. Email: ellison@math.unm.edu

Received: 19 May 1993/in revised form: 11 March 1994

Abstract: Let I be a set of invariants and θ be a set of angle variables for a system of differential equations with an $O(\varepsilon)$ vector field. When time dependent stochastic perturbations, also of $O(\varepsilon)$, are added to the system, we have shown that under suitable conditions I becomes a stochastic adiabatic invariant satisfying a diffusion equation on time scales of order $1/\varepsilon^2$, in the limit as $\varepsilon \to 0$. Here we show that the angle variables converge weakly to a Gaussian Markov process on an $O(\varepsilon^{-4/3})$ time scale, and thus the phase becomes randomized at these times. Application to nearly integrable Hamiltonian systems is considered.

0. Introduction

We consider the behavior of the stochastic differential equation in $\mathbb{R}^d$,

$$\dot{x} = \varepsilon f(x,t) + \varepsilon F(x,t,\omega) + o(\varepsilon^{5/3}) \tag{0.1}$$

as $\varepsilon \to 0$. We require that the expectation $EF(x,t) = 0$ and that the time average

$$\bar{f}(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(x,s)\,ds \tag{0.2}$$

exists for all x.

Making the change of scale $v = \varepsilon t$, (0.1) becomes

$$\frac{dx}{dv} = f\left(x,\frac{v}{\varepsilon}\right) + F\left(x,\frac{v}{\varepsilon},\omega\right) + o(\varepsilon^{2/3})\,. \tag{0.3}$$

Then (0.2) and the law of large numbers applied to F suggest that the method of averaging may apply to (0.3), and for small ε the solution should be close to the "unperturbed equation"

$$\frac{dx}{dv} = \bar{f}(x)\,. \tag{0.4}$$

★ Supported by NSF grant DMR-8704348