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A Nontrivial Renormalization Group Fixed Point for the Dyson–Baker Hierarchical Model

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Abstract: We prove the existence of a nontrivial Renormalization Group (RG) fixed point for the Dyson–Baker hierarchical model in $d = 3$ dimensions. The single spin distribution of the fixed point is shown to be entire analytic, and bounded by $\exp(-\mathrm{const} \times t^6)$ for large real values of the spin t. Our proof is based on estimates for the zeros of a RG fixed point for Gallavotti's hierarchical model. We also present some general results for the heat flow on a space of entire functions, including an order preserving property for zeros, which is used in the RG analysis.

1. Introduction and Main Results

One of the basic assumptions in the modern theory of critical phenomena is the existence of nontrivial renormalization group (RG) fixed points, associated with certain universality classes of interactions. Within the framework of statistical mechanics, this assumption is actually a conjecture, and it should be possible to either prove it or disprove it. However, even for the simplest classes of "realistic" interactions, such as the one represented by the three–dimensional Ising model, the rigorous construction of a nontrivial RG fixed point seems beyond the reach of presently known methods. In addition, there is a lack of good numerical results in this area, which indicates that even at a quite fundamental level there are still gaps in our understanding of RG transformations.

The traditional approach in such situations is to try to first understand some simpler, and thus necessarily less realistic, class of interactions. In this case, the model with the longest history is Dyson's hierarchical model [1–4]. For the Dyson–type hierarchical analogue of the nearest–neighbor (continuous spin) Ising model in $d = 3$ dimensions, which was first considered by Baker [3], the full RG transformation $\mathscr{T}$ reduces to the third power of the following nonlinear operator $\mathscr{R}$:

$$\big(\mathscr{R}(h)\big)(t) = K \int_{-\infty}^{\infty} dx\, e^{-2\sigma x^2} h(\alpha t + x) h(\alpha t - x)\,, \qquad t \in \mathbb{R}\,. \tag{1.1}$$