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Zero Measure Spectrum for the Almost Mathieu Operator

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Abstract: We study the almost Mathieu operator: $(H_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n)$, on $l^2(Z)$, and show that for all λ, θ, and (Lebesgue) a.e. α, the Lebesgue measure of its spectrum is precisely $|4 - 2|\lambda||$. In particular, for $|\lambda| = 2$ the spectrum is a zero measure cantor set. Moreover, for a large set of irrational α's (and $|\lambda| = 2$) we show that the Hausdorff dimension of the spectrum is smaller than or equal to 1/2.

1. Introduction

In this paper, we study the almost Mathieu (also called Harper's) operator on $l^2(Z)$. This is the (bounded, self adjoint) operator $H_{\alpha,\lambda,\theta}$, defined by:

$$\begin{aligned} H_{\alpha,\lambda,\theta} = H_0 + V_{\alpha,\lambda,\theta}\,, \qquad (H_0u)(n) &= u(n+1) + u(n-1)\,, \\ (V_{\alpha,\lambda,\theta}u)(n) &= \lambda\cos(2\pi\alpha n + \theta)u(n)\,, \end{aligned} \tag{1.1}$$

where $\alpha, \lambda, \theta \in R$.

$H_{\alpha,\lambda,\theta}$ is a tight binding model for the Hamiltonian of an electron in a one dimensional lattice, subject to a commensurate (if α is rational) or incommensurate (if α is irrational) potential. It is also related to the Hamiltonian of an electron in a two dimensional lattice, subject to a perpendicular magnetic field [11, 13] (in which case the relevant energy spectrum is the union over θ of the energy spectra of $H_{\alpha,\lambda,\theta}$).

The almost Mathieu operator has been studied by many authors [1–13, 15, 17–24, 26], and many of its spectral characteristics are known. Our main result in this paper is:

Theorem 1. *If α is an irrational, for which there is a sequence of rationals $\{p_n/q_n\}$ obeying:*

$$\lim_{n\to\infty} {q_n}^2 \left|\alpha - \frac{p_n}{q_n}\right| = 0\,,$$

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