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Conformal Blocks and Generalized Theta Functions

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Abstract: Let $\mathscr{S}\mathscr{U}_X(r)$ be the moduli space of rank r vector bundles with trivial determinant on a Riemann surface X. This space carries a natural line bundle, the determinant line bundle $\mathscr{L}$. We describe a canonical isomorphism of the space of global sections of $\mathscr{L}^k$ with the space of conformal blocks defined in terms of representations of the Lie algebra $\mathfrak{sl}_r(\mathbf{C}((z)))$. It follows in particular that the dimension of $H^0(\mathscr{S}\mathscr{U}_X(r), \mathscr{L}^k)$ is given by the Verlinde formula.

Introduction

The aim of this paper is to construct a canonical isomorphism between two vector spaces associated to a Riemann surface X. The first of these spaces is the space of *conformal blocks* $B_c(r)$ (also called the space of vacua), which plays an important role in conformal field theory. It is defined as follows: choose a point $p \in X$, and let A_X be the ring of algebraic functions on $X - p$. To each integer $c \geq 0$ is associated a representation V_c of the Lie algebra $\mathfrak{sl}_r(\mathbf{C}(z))$), the *basic representation* of level c (more correctly it is a representation of the universal extension of $\mathfrak{sl}_r(\mathbf{C}((z)))$ – see Sect. 7 for details). The ring A_X embeds into $\mathbf{C}((z))$ by associating to a function its Laurent development at p; then $B_c(r)$ is the space of linear forms on V_c which vanish on the elements $A(z)v$ for $A(z) \in \mathfrak{sl}_r(A_X)$, $v \in V_c$.

The second space comes from algebraic geometry, and is defined as follows. Let $\mathscr{S}\mathscr{U}_X(r)$ be the moduli space of semi-stable rank r vector bundles on X with trivial determinant. One can define a theta divisor on $\mathscr{S}\mathscr{U}_X(r)$ in the same way one does in the rank 1 case: one chooses a line bundle L on X of degree $g-1$, and considers the locus of vector bundles $E \in \mathscr{S}\mathscr{U}_X(r)$ such that $E \otimes L$ has a nonzero section. The associated line bundle $\mathscr{L}$ is called the *determinant bundle*; the space we are interested in is $H^0(\mathscr{S}\mathscr{U}_X(r), \mathscr{L}^c)$. This space can be considered as a non-Abelian version of the

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