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Solvability of the Localized Induction Equation for Vortex Motion

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Abstract: The initial and the initial-boundary value problems for the localized induction equation which describes the motion of a vortex filament are considered. We prove the existence of solutions of both problems globally in time in the sense of distribution by the method of regularization.

1. Introduction

The localized induction equation which describes the motion of a smooth thin vortex filament in three-dimensional perfect fluid is derived from some physical approximations of the Biot–Savart law ([1, 4]). It is formulated as

$$\mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss} , \tag{1.1}$$

where $\mathbf{x} = \mathbf{x}(s, t)$ denotes the coordinate of a point on the filament in $\mathbb{R}^3$ as a vector-valued function of arclength $s \in \mathbb{R}$ and time t, and the subscripts mean the partial differentiation with respect to the corresponding variables.

Some exact solutions of (1.1) are known ([7]): the trivial type ($\mathbf{x}_s \times \mathbf{x}_{ss} = 0$), the circular and the helical ones ($|\mathbf{x}_s \times \mathbf{x}_{ss}| = \text{const.}$), the elastic one rotating about an axis without changing its own form, etc.

Besides, Hasimoto indicated in [5] that (1.1) can be transformed by means of the Frenet–Serret formulae into the nonlinear Schrödinger equation,

$$-i\Psi_t = \Psi_{ss} + (1/2)|\Psi|^2\Psi \tag{1.2}$$

for $\Psi = \kappa(s, t)\exp\{i\int_0^s \tau(s,t)\,ds - i\int_0^t a(t)/2\,dt\}$. Here κ and τ are the curvature and the torsion of the filament respectively, i.e.,

$$\kappa = |\mathbf{x}_{ss}|, \quad \tau = \mathbf{x}_s \cdot (\mathbf{x}_{ss} \times \mathbf{x}_{sss})/|\mathbf{x}_{ss}|^2 ,$$

and