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Existence Theorem for Solitary Waves on Lattices

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Abstract: In this article we give an existence theorem for localized travelling wave solutions on one-dimensional lattices with Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \left(\tfrac{1}{2}\, p_n^2 + V(q_{n+1} - q_n)\right),$$

where $V(\cdot)$ is the potential energy due to nearest-neighbour interactions. Until now, apart from rare integrable lattices like the Toda lattice $V(\phi) = ab^{-1}(e^{-b\phi} + b\phi - 1)$, the only evidence for existence of such solutions has been numerical. Our result in particular recovers existence of solitary waves in the Toda lattice, establishes for the first time existence of solitary waves in the (nonintegrable) cubic and quartic lattices $V(\phi) = \frac{1}{2}\,\phi^2 + \frac{1}{3}\,a\phi^3$, $V(\phi) = \frac{1}{2}\,\phi^2 + \frac{1}{4}\,b\phi^4$, thereby confirming the numerical findings in [1] and shedding new light on the recurrence phenomena in these systems observed first by Fermi, Pasta and Ulam [2], and shows that contrary to widespread belief, the presence of exact solitary waves is not a peculiarity of integrable systems, but "generic" in this class of nonlinear lattices. The approach presented here is new and quite general, and should also be applicable to other forms of lattice equations: the travelling waves are sought as minimisers of a naturally associated variational problem (obtained via Hamilton's principle), and existence of minimisers is then established using modern methods in the calculus of variations (the concentration-compactness principle of P.-L. Lions [3]).

1. Introduction

Discrete Hamiltonian systems arising in mechanics and physics are notoriously less amenable to analytic techniques than their continuous counterparts. Thus the majority of work on solitary waves on lattices has centred on two areas. Firstly continuum approximations; here the equation of motion reduces to a PDE [4–7] (and hence the

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