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The Non-Linear Stability of Front Solutions for Parabolic Partial Differential Equations

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Abstract: For the Ginzburg-Landau equation and similar reaction-diffusion equations on the line, we show convergence of *complex* perturbations of front solutions towards the front solutions, by exhibiting a coercive functional.

1. Introduction and Statement of Results

In this paper, we study partial differential equations of the form

$$\partial_t u = \partial_x^2 u + uF(|u|) , \tag{1.1}$$

$u = u(x,t)$, with $t > 0$, $x \in \mathbf{R}$, and u taking complex values. We assume $F(0) > 0$, $F(a) = 0$, for $a > 0$, and without loss of generality we consider only the case $a = 1$. A front solution of (1.1) is a solution u of the form $u(x,t) = f(x - ct) \in \mathbf{R}$, $c > 0$, with $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = 1$. The most studied equation of this type is the Ginzburg-Landau (GL) equation (or Newell-Whitehead equation) where $F(\zeta) = 1-\zeta^2$.

Our aim is to study the stability of such fronts for initial data u_0 which are small, *complex* perturbations of the front f of the form

$$u_0(x) = f(x)\big(1 + r_0(x)\big)e^{i\varphi_0(x)} . \tag{1.2}$$

We will also write

$$u(x,t) = f(x - ct)\big(1 + r_t(x - ct)\big)e^{i\varphi_t(x-ct)} , \tag{1.3}$$

and it is always tacitly assumed that $u(x,t)$ solves Eq. (1.1). Note that both r and φ are measured in the frame in which the front itself moves. Complex perturbations of the real front solutions occur naturally in "amplitude equations" such as in the reduction from the Swift-Hohenberg equation to the Ginzburg-Landau equation, see, e.g., [CE1].

We will give sufficient conditions on F and c to insure that solutions of Eq. (1.1) with initial conditions as in Eq. (1.2) *converge* to the front solution, provided r_0