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The Behaviour of Eigenstates of Arithmetic Hyperbolic Manifolds

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Abstract. In this paper we study some problems arising from the theory of Quantum Chaos, in the context of arithmetic hyperbolic manifolds. We show that there is no strong localization ("scarring") onto totally geodesic submanifolds. Arithmetic examples are given, which show that the random wave model for eigenstates does not apply universally in 3 degrees of freedom.

1. Introduction

Let $X = \Gamma\backslash\mathbf{H}^2$ be a compact hyperbolic surface, with $\Gamma \subset PSL(2, \mathbf{R})$ a discrete co-compact subgroup, and $\mathbf{H}^2$ the hyperbolic plane. As is well known, the geodesic flow on the unit cotangent bundle S^*X is ergodic, Anosov and displays chaotic features [1]. Denote by Δ the Laplace-Beltrami operator for X, and by λ_j (respectively ϕ_j) its eigenvalues (resp. an orthonormal basis of eigenstates). Δ is the quantization of the Hamiltonian generating the geodesic flow. For this reason the behaviour of λ_j and ϕ_j has been studied extensively in the context of Quantum Chaos. One of the central questions is whether the ϕ_j's behave like random waves, or if they display some localization or other structure related to the classical trajectories. In the case of the billiards in the Bunimovich stadium (which is a somewhat more complicated chaotic system), Heller [11] unexpectedly found that certain states are enhanced on a finite union of periodic (unstable) orbits. He called this phenomenon "scarring." In our case, the numerical evidence [9, 2, 3] points to the eigenstates behaving like random waves. We refer to Heller [11] for a discussion of random waves.

In this paper we examine these issues for Γ arithmetic, in fact for certain congruence subgroups derived from quaternion algebras. These are of course special, but in connection with the questions at hand we expect that our results are typical of

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