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Callias' Index Theorem, Elliptic Boundary Conditions, and Cutting and Gluing

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Abstract: It is shown that elliptic boundary conditions play the same role in Callias' index theorem as spectral boundary conditions do in the Atiyah–Patodi–Singer index theorem. This is used to generalize Callias' index theorem to arbitrary complete spin-manifolds.

1. The Index Formula

Let X be a complete odd-dimensional smooth oriented spin-manifold, with complex spinor bundle S. Let V be a smooth Hermitian vector bundle over X, with a smooth unitary connection A and a smooth Hermitian endomorphism Φ. Let $\partial\!\!\!/_A$ denote the coupled Dirac operator acting on sections of $S \otimes V$. Form the operators

$$D = \partial\!\!\!/_A + i1 \otimes \Phi$$

and

$$D^* = \partial\!\!\!/_A - i1 \otimes \Phi$$

acting on sections of $S \otimes V$.

The index problem for such operators was first studied by C. Callias [C], who proved an index theorem in the case $X = \mathbb{R}^{2n+1}$, using traces of integral kernels. Callias' index theorem can also be derived from Fedosov's index theorem for elliptic operators on Euclidean space [F], see also Sect. 19.3 in [H2], as explained in [BS], see also [A1]. Callias' index theorem was generalized by N. Anghel [A2] to Dirac operators coupled to the trivial Hermitian vector bundle over manifolds with warped cylindrical ends, using the relative index theory of [GL]. In this paper we generalize this index theorem to Dirac operators on arbitrary complete oriented spin-manifolds coupled to arbitrary Hermitian vector bundles.

Let $\lambda(x)$ denote the smallest of the absolute values of the eigenvalues of $\Phi(x)$. Let $\hat{A}$ denote the $\hat{A}$-genus.