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Nahm's Equations and Hyperkähler Geometry

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Abstract. The geometry of certain moduli spaces of solutions to Nahm's equations is studied, and a family of gravitational instantons is shown to arise as a deformation of the Atiyah-Hitchin manifold.

1. Introduction

Considerable effort has been devoted to the study of moduli spaces of solutions to the self-dual Yang-Mills equations and their dimensional reductions. One reason for this is that such moduli spaces can often be naturally endowed with a hyperkähler structure. This consists of a metric and three covariant constant complex structures satisfying the quaternionic multiplication relations. Hyperkähler manifolds are necessarily $4n$-dimensional, where n is an integer, and their holonomy is contained in $Sp(n)$. The possible existence of such manifolds was implicit in Berger's classification of the groups which could arise as holonomy groups of non-symmetric Riemannian manifolds: however nontrivial examples of dimension higher than four were not known until the work of Calabi [C]. Four-dimensional hyperkähler manifolds are, in the terminology of physics, examples of gravitational instantons.

In this paper, we shall introduce a twelve-dimensional moduli space M^{12} of solutions to Nahm's equations, a nonlinear system of ordinary differential equations arising as a reduction of the self-dual Yang-Mills equations. The manifold M^{12} admits a hyperkähler structure, and is acted on isometrically by $U(2)$ and Spin(3). The $U(2)$ action is triholomorphic (preserves the Kähler structures) while the action of Spin(3) permutes the Kähler structures. The hyperkähler quotient of M^{12} by the centre of $U(2)$ is an eight-dimensional hyperkähler manifold M^8 with an isometric $SU(2) \times SO(3)$ action. We show that M^8 is homeomorphic to $\mathbb{R}^5 \times SU(2)/\mathbf{Z}_2$ and calculate the L^2 metric on M^8 and on the quotient $N^5 = M^8/SU(2)$. We also study a totally geodesic submanifold Σ of M^8 which represents axisymmetric solutions to the Nahm equations. Finally, we obtain a family of hyperkähler four-manifolds as hyperkähler quotients of M^8 by a circle subgroup of $SU(2)$. These manifolds, which we believe