

A Self-Dual Yang–Mills Hierarchy and its Reductions to Integrable Systems in $1 + 1$ and $2 + 1$ Dimensions

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Abstract. The self-dual Yang–Mills equations play a central role in the study of integrable systems. In this paper we develop a formalism for deriving a four dimensional integrable hierarchy of commuting nonlinear flows containing the self-dual Yang–Mills flow as the first member. We show that upon appropriate reduction and suitable choice of gauge group it produces virtually all well known hierarchies of soliton equations in $1 + 1$ and $2 + 1$ dimensions and can be considered as a “universal” integrable hierarchy. Prototypical examples of reductions to classical soliton equations are presented and related issues such as recursion operators, symmetries, and conservation laws are discussed.

1. Introduction

In recent years there has been considerable interest in the self-dual Yang–Mills (SDYM) equations. Originated from the non-perturbative approach to the quantum theory of gauge fields [1], SDYM system became useful also in general relativity [2]; in mathematics it was a major tool for achieving dramatic progress in the theory of four-manifolds [3]. Moreover, SDYM equations turn out to be an integrable system in the sense that associated with them there exists a compatible pair of linear PDE's from which significant information about the SDYM equations can be deduced, e.g. solutions via the Riemann–Hilbert factorization problem, Bäcklund transformations, conservation laws, hierarchies, etc. [4, a, b, c].

More recently, many well known soliton equations solvable by the inverse scattering transform (IST) have been found as reductions of the SDYM system which has led to the conjecture that perhaps it contains “all” integrable soliton systems [5]. Indeed by extending the notion of SDYM system to encompass the higher flows we give more credence to this possibility. In particular, we show that the SDYM equations can be embedded naturally in a hierarchy of integrable equations of which they can be considered as the principal starting member. Significantly, the classical soliton hierarchies are constructed in a similar way and to stress this point further we show how the resulting four dimensional “universal” hierarchy reduces to the standard $1 + 1$ dimensional soliton hierarchies. In order