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Monopoles, Braid Groups, and the Dirac Operator

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Abstract. Using the relation between the space of rational functions on $\mathbb{C}$, the space of $SU(2)$-monopoles on $\mathbb{R}^3$, and the classifying space of the braid group, see [10], we show how the index bundle of the family of real Dirac operators coupled to $SU(2)$-monopoles can be described using permutation representations of Artin's braid groups. We also show how this implies the existence of a pair consisting of a gauge field A and a Higgs field Φ on $\mathbb{R}^3$ whose corresponding Dirac equation has an arbitrarily large dimensional space of solutions.

1. Introduction and Statement of Results

Let $\mathcal{M}_k$ denote the space of based, $SU(2)$ monopoles in $\mathbb{R}^3$ of charge k. Thus an element of $\mathcal{M}_k$ is represented by a configuration (A, Φ), where A, the gauge field, is a smooth connection on the trivial $SU(2)$ bundle P over $\mathbb{R}^3$ and Φ, the Higgs field, is a smooth section of the vector bundle associated to P via the adjoint representation. Since the bundle P is trivial A can be identified with a smooth 1-form on $\mathbb{R}^3$ with values in the Lie algebra $\mathfrak{su}(2)$ and Φ can be identified with a smooth function $\Phi\colon \mathbb{R}^3 \to \mathfrak{su}(2)$. We equip $\mathbb{R}^3$ with its standard metric and orientation and $\mathfrak{su}(2)$ with its standard invariant inner product. The pair (A, Φ) is a monopole if it satisfies the following conditions:

(1) $1 - |\Phi| \in L^6(\mathbb{R}^3)$.

(2) The pair (A, Φ) has finite energy; that is the Yang–Mills–Higgs functional is finite

$$\mathcal{U}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|F_A|^2 + |D_A \Phi|^2)\, \mathrm{dvol} < \infty .$$

Here D_A is the covariant derivative operator defined by A and F_A is the curvature of A.

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