Commun. Math. Phys. 158, 93–125 (1993)

Communications in
Mathematical
Physics

Weyl's Problem for the Spectral Distribution of Laplacians on P.C.F. Self-Similar Fractals

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Received June 22, 1992; in revised form January 25, 1993

Abstract. We establish an analogue of Weyl's classical theorem for the asymptotics of eigenvalues of Laplacians on a finitely ramified (i.e., p.c.f.) self-similar fractal K, such as, for example, the Sierpinski gasket. We consider both Dirichlet and Neumann boundary conditions, as well as Laplacians associated with Bernoulli-type ("multifractal") measures on K. From a physical point of view, we study the density of states for diffusions or for wave propagation in fractal media. More precisely, let $\varrho(x)$ be the number of eigenvalues less than x. Then we show that $\varrho(x)$ is of the order of $x^{d_S/2}$ as $x \to +\infty$, where the "spectral exponent" d_S is computed in terms of the geometric as well as analytic structures of K. Further, we give an effective condition that guarantees the existence of the limit of $x^{-d_S/2}\varrho(x)$ as $x \to +\infty$; this condition is, in some sense, "generic". In addition, we define in terms of the above "spectral exponents" and calculate explicitly the "spectral dimension" of K.

0. Introduction

In this paper, we will study the asymptotic behavior of the spectrum of the Laplacians on some self-similar sets. This problem occurs naturally in the study of physical phenomena, such as waves and diffusions, on fractal objects.

At first, we recall Weyl's classical result. Let Ω be a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$. We consider the following eigenvalue problem:

$$\text{(DE)} \begin{cases} \Delta u = -ku \text{ on } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplacian on $\mathbb{R}^n$. It is well known that the eigenvalues – i.e., the scalars k such that (DE) has a non-trivial solution u – are non-negative and

* The research of Michel L. Lapidus was partially supported by the National Science Foundation under Grant DMS-8904389 (transferred to the University of California as grant DMS-9196085) and Grant DMS-9207098, as well as by Yale University