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Difference Almost-Periodic Schrödinger Operators: Corollaries of Localization

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Abstract. We study the technique used in proving the exponential localization for one-dimensional difference Schrödinger operators with quasi-periodic potential. In this way we get some corollaries concerning the spectrum structure near the boundaries and the existence of bounded, non-exponentially decaying solutions of the equation on eigenvalues.

1. Introduction

In this paper we study the family of Schrödinger operators, acting in $\ell^2(\mathbb{Z}^1)$ as follows:

$$(\mathbf{H}_\varepsilon(\alpha)\psi)(n)=\varepsilon\cdot(\psi(n-1)+\psi(n+1))+\mathbf{V}(\alpha+n\cdot\omega)\cdot\psi(n)\,, \tag{1.1}$$

with $\alpha\in\mathbf{S}^1$, $\omega\in\mathbf{R}^1$, $\mathbf{V}\in\mathbf{C}^2(\mathbf{S}^1)$, and ε being sufficiently small. One can consider $\mathbf{H}_\varepsilon(\alpha)$ as a metrically transitive operator in the sense of [5]. Indeed, $\mathbf{H}_\varepsilon(\mathbf{T}\alpha)=\mathbf{U}^{-1}\mathbf{H}_\varepsilon(\alpha)\mathbf{U}$, where $\mathbf{T}\alpha=(\alpha+\omega)\,(\mathrm{mod}\,1)$ and $\mathbf{U}$ is a unitary shift operator: $(\mathbf{U}f)(n)=f(n-1)$. For $\mathbf{V}(\alpha)=\cos(2\pi\alpha)$ we get the Almost-Mathieu operator as an important particular case. We shall use the results for (1.1) obtained by Sinai [1] and Fröhlich, Spencer, and Wittwer [3, 4]:

Main Theorem (Sinai, Fröhlich, Spencer, and Wittwer). *Let $\mathbf{V}\in\mathbf{C}^2(\mathbf{S}^1)$ have exactly two critical points, both being non-degenerate. Let $\omega\in[0;1]$ be a Diophantine number, i.e. a number, satisfying the condition $|\omega-p/q|\geqq\mathrm{const}\cdot q^{-\delta-2}$ for some $\delta>0$. Then there exists a positive number $\varepsilon_0=\varepsilon_0(\mathbf{V},\delta)$ such that for any ε, $|\varepsilon|<\varepsilon_0$ and a.e. $\alpha\in\mathbf{S}^1$ the operator $\mathbf{H}_\varepsilon(\alpha)$ has purely point spectrum. All its eigenfunctions decay exponentially. The support of the density of states for $\mathbf{H}_\varepsilon(\alpha)$ is a nowhere dense Cantor set of positive Lebesgue measure and the total Lebesgue measure of all spectral gaps for $\mathbf{H}_\varepsilon(\alpha)$ is less than* const $\cdot|\varepsilon|$.

We shall derive in this paper several corollaries from the Main Theorem:

Corollary 1. *For all $\varepsilon:|\varepsilon|<\varepsilon_0$ there exists a countable set $\mathscr{A}(\varepsilon)\subset S^1$ such that for every $\alpha\in\mathscr{A}$: $\lambda_{\max}=\sup\{\lambda\colon\lambda\in\mathrm{Sp}\,\mathbf{H}_\varepsilon(\alpha)\}$ is an eigenvalue of $\mathbf{H}_\varepsilon(\alpha)$.*