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Non-linear Wave and Schrödinger Equations

I. Instability of Periodic and Quasiperiodic Solutions*

I. M. Sigal**

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada

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Abstract. We investigate stability of periodic and quasiperiodic solutions of linear wave and Schrödinger equations under non-linear perturbations. We show in the case of the wave equations that such solutions are unstable for generic perturbations. For the Schrödinger equations periodic solutions are stable while the quasiperiodic ones are not. We extend these results to periodic solutions of non-linear equations.

1. Introduction

It is well known that the world is non-linear. However, most of our knowledge about it is derived from analysis of its linear approximations. Though non-linear perturbations are usually extremely weak, they can alter the linear behaviour qualitatively. Thus it is important to understand how the most elementary and fundamental properties of linear systems are affected by non-linear perturbations.

Consider problems concerning the time evolution. Once existence of solutions is established the next goal here is classification of the orbits (= solutions) w.r. to their localization in the configuration space of the system in question, namely, into bounded and unbounded. In the case of linear Schrödinger and wave equations the Ruelle theorem allows us to identify bounded orbits with periodic and quasiperiodic (in time) ones, produced by eigenfunctions of the Schrödinger or wave operator involved, and their linear combinations. Thus the problem: investigate stability of the (quasi) periodic solutions of the linear equations under non-linear perturbations. This problem was posed by J. Fröhlich and T. Spencer several years ago and is the subject of the present paper.

In this paper we show that periodic and quasiperiodic solutions of the linear wave equation are unstable under generic non-linear perturbations. For the Schrödinger equation some of the periodic solutions are stable while the others as well as certain quasiperiodic solutions are not.

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