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An Algebraic Approach to the Planar Coloring Problem

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Abstract. We point out a general relationship between the planar coloring problem with Q colors and the Temperley-Lieb algebra with parameter $\sqrt{Q}$. This allows us to give a complete algebraic reformulation of the four color result, and to give algebraic interpretations of various other aspects of planar colorings.

Introduction

The purpose of this paper is to delineate the relationship of the Temperley-Lieb algebra [TL] with planar graph coloring problems. The main result is a complete algebraic reformulation of the four color theorem [AH]. This reformulation is a special case of a simply stated and more general conjecture about the Temperley-Lieb algebra.

The paper is organized as follows. In the first section we recall the definition of the chromatic polynomial, the dichromatic polynomial, and of the Potts model. In the second section we recall the definition of the Temperley-Lieb algebra and of the Potts model representation. In Sect. Three we prove our first non-trivial result (Proposition 3.1). It states that the Potts model partition function (hence in particular the chromatic polynomial) for any planar graph can be written as the trace of a "transfer matrix" (Definition 3.2), a well defined product of elementary edge operators (Definition 2.3) in the Temperley-Lieb algebra. Such a result was known so far for regular lattices only. In Sect. Four we discuss some properties of the transfer matrices. We show in particular the reciprocal of Proposition 3.1 (Proposition 4.1), namely that an arbitrary product of edge operators can be

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