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Quantum Invariants at the Sixth Root of Unity

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Abstract. A general topological formula is given for the $SU(2)$ quantum invariant of a 3-manifold M at the sixth root of unity. It is expressed in terms of the homology, Witt invariants and signature defects of the various 2-fold covers of M, and thus ties in with basic 4-dimensional invariants. A discussion of the range of values of these quantum invariants is included, and explicit evaluations are made for lens spaces.

Introduction

Quantum invariants of 3-manifolds were introduced by Witten in 1988 using Chern–Simons gauge theory and path integrals [W], and subsequently formulated in terms of quantum groups by Reshetikhin and Turaev [RT]. They depend on the choice of a simple compact Lie group (the *gauge group*) and a root of unity q (of order three or more).

In [KM2], the first two authors established a cabling formula and a symmetry principle for link invariants derived from quantum groups which led to an elementary proof of the existence of the quantum invariants for an $SU(2)$ gauge and to evaluations at the third and fourth roots of unity (in terms of algebraic topological invariants). The existence of such evaluations is not surprising in light of the fact that the $SU(2)$ quantum invariants at q of a 3-manifold M_L obtained by surgery on a framed link L in S^3 are related to the values of the Jones polynomial of L at q, and these values are understood topologically for q of order three or four. In fact, they are also understood at the sixth root of unity, and so it is natural to look for a corresponding evaluation of the quantum invariants. This has been found for 3-manifolds obtained by surgery on a single knot [KM1]. The purpose of this paper is to give a formula for arbitrary closed oriented 3-manifolds M. We adopt the notation of [KM2]. In particular, $\tau_r(M)$ denotes the $SU(2)$ quantum invariant of M at $q = \exp(2\pi i/r)$.

Recall from [KM2, Sect. 8.32] that for $r \equiv 2 \pmod 4$, $\tau_r(M)$ splits as a sum of invariants $\tau_r(M, \Theta)$ of M equipped with a 1-dimensional cohomology class Θ. (This