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Factorization of Random Jacobi Operators and Bäcklund Transformations

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Summary. We show that a positive definite random Jacobi operator L over an abstract dynamical system $T : X \to X$ can be factorized as $L = D^2$, where D is again a random Jacobi operator but defined over a new dynamical system $S : Y \to Y$ which is an integral extension of T. An isospectral random Toda deformation of L corresponds to an isospectral random Volterra deformation of D. The factorization leads to commuting Bäcklund transformations which can be written explicitly in terms of Titchmarsh-Weyl functions. In the periodic case, the Bäcklund transformations are time 1 maps of a Toda flow with a time dependent Hamiltonian.

1. Introduction

Bäcklund transformations for Toda lattices have been given in a nonexplicit form by Toda and Wedati [WT,T]. Adler [A] found that Bäcklund transformations have their origin in a factorization $L = AA^*$ in analogy to the Miura map for the KdV equation. It has been mentioned already by Moser [M] that the relation between the Kac v. Moerbeke system and the Toda lattice has its algebraic origin in a factorization $L = D^2$, where D is a matrix on a vector space with twice the dimension of the vector space on which L acts. In those papers D or A are given first and L is obtained by forming $L = AA^* = D^2$. Recently, the Poisson structure of the Bäcklund transformations was studied in [DL] for the periodic Toda lattice and also in the more general context of Toda equations on Lie groups.

In [K] we studied Toda lattices with random boundary conditions. They were obtained by making isospectral deformations of random Jacobi operators. The random Toda lattice is a generalization of both the periodic and the tied Toda lattice. It is defined over an arbitrary abstract dynamical system. We will show here that Bäcklund transformations can also be done in this case. They are generalizing the Bäcklund transformations known for periodic and aperiodic Toda lattices investigated in [T,A,DL]. What is new here, (beside the fact that we are working with random