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Instantons and Representations of an Associative Algebra

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Abstract. We give the correspondence between instantons on S^4 and some representations of an associative algebra. For the given structure group, we get simultaneous imbeddings to $\mathbb{C}^\infty$ (the inductive limit) of the moduli spaces for instantons on S^4 of all instanton numbers.

In this note we show that instantons on S^4 can be identified with some representations of an associative algebra.

Let A be the free algebra over $\mathbb{C}$ generated by two elements q, p. We define a new multiplication $*$ in A as follows:

$$f_1 * f_2 = f_1(pq - qp)f_2, \quad f_1, f_2 \in A .$$

Then $(A, *)$ is an associative algebra (with no unit), which is an extention of the Weyl algebra $\mathbb{C}\left[q, \frac{d}{dq}\right]$. We consider finite dimensional representations of $(A, *)$. Let W be the complex vector space of dimension l, and h be a linear map from A to End W. Then h induces a linear map $\tilde{h}$: $A \otimes W \to A^* \otimes W$ defined by

$$\langle \tilde{h}(f_1 \otimes w), f_2 \rangle = h(f_2 f_1)w, \quad f_1, f_2 \in A, \ w \in W .$$

We denote by $H(l, k)$ the set of all algebra homomorphisms $h: (A, *) \to \text{End } W$ such that the rank of $\tilde{h}$ is k. If h is an algebra homomorphism from $(A, *)$ to End W, then

$$h(f_1(pq - qp)f_2) = h(f_1)h(f_2) ,$$

so the linear map h is determined by $h(q^j p^i)$, $i, j \geqq 0$.

Let P be the principal $SU(l)$ bundle over $S^4 = \mathbb{R}^4 \cup \infty$ with $c_2 = k$, and $\tilde{M}(SU(l), k)$ be the framed moduli space for anti-self-dual (ASD) connections on P: $\{$ASD connections on $P\}/\mathcal{G}_\infty$, where $\mathcal{G}_\infty$ stands for the group of all gauge transformations on P fixing the points in the fiber over ∞. $\tilde{M}(SU(l), k)$ is a $4kl$-dimensional smooth manifold [1].