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An Asymptotic Expression for the Splitting of Separatrices of the Rapidly Forced Pendulum

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Abstract. The measure of the splitting of the separatrices of the rapidly forced pendulum

$$\ddot{x} + \sin x = \mu \sin \frac{t}{\varepsilon},$$

is considered as a model problem that has been studied by different authors. Here ε, μ are small parameters, $\varepsilon > 0$, but otherwise independent. The following formula for the angle α between separatrices is established

$$\alpha = \frac{\pi}{2\varepsilon} \frac{\mu}{\cosh \dfrac{\pi}{2\varepsilon}} [1 + O(\mu, \varepsilon^2)] .$$

This formula is also valid for the particular case $\mu = \varepsilon^p$, with $p > 0$, $\varepsilon > 0$, and agrees with the one provided by the first order Poincaré–Melnikov theory that cannot be applied directly, due to the exponentially small dependence of α on the parameter ε.

1. Introduction

Let us consider the equation of the rapidly forced pendulum

$$\ddot{x} + \sin x = \mu \sin \frac{t}{\varepsilon}, \tag{1.1}$$

where ε, μ are small parameters, $0 < \varepsilon < 1$, but otherwise independent. This equation can be considered as a model of a two-dimensional integrable system perturbed by a very rapidly oscillatory forcing. Also, performing the change of time $\tau = t/\varepsilon$, it can be considered as a nearly integrable system with slow dynamics:

$$x'' + \varepsilon^2 \sin x = \mu \varepsilon^2 \sin \tau \quad \left(' = \frac{d}{d\tau} \right). \tag{1.2}$$