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The Ground State Energy of Schrödinger Operators

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Abstract. We study $e(\lambda) = \inf \operatorname{spec}(-\Delta + \lambda V)$ and examine when $e(\lambda) < 0$ for all $\lambda \neq 0$. We prove that $-c\lambda^2 \leqq e(\lambda) \leqq -d\lambda^2$ for suitable V and all small $|\lambda|$.

1. Introduction

In this paper we want to look at the "ground state energy," $e(\lambda) = \inf \operatorname{spec}(-\Delta + \lambda V)$, of a Schrödinger operator $-\Delta + \lambda V$ for V's which do *not* decay at infinity – think of periodic or almost periodic problems. In particular, we want to see when $e(\lambda)$ is *strictly* negative for all $\lambda \neq 0$. There is a large literature on this problem and the weaker $e \leqq 0$ result, most of it in one dimension. These examples typically have only essential spectrum so $e(\lambda) < 0$ is equivalent to solutions of $-u'' + \lambda V u = 0$ having an infinite number of zeros. The one-dimensional results often are phrased in these terms ("$-d^2/dx^2 + \lambda V$ is oscillatory").

The earliest results we are aware of are those of Wintner [19], who studied $\frac{-d^2}{dx^2} + \lambda V$ with $V(x+1) = V(x)$. He showed that

$$\lambda \int_0^1 V(x)dx - C\lambda^2 \int_0^1 V^2(x)dx \leqq e(\lambda) \leqq \lambda \int_0^1 V(x)dx \tag{1.1}$$

holds with $C = 1$. Kato [8] then improved this to $C = 1/16$. The question about the optimal C has been raised in [6, 8, 12, 19]. In Sect. 6 we will show that $C = (2\pi)^{-2}$ is best possible, the first inequality in (1.1) being strict for $\lambda \neq 0$.

In Sect. 5 we will recover Kato's result.

A series of authors (Moore [11], Blumenson [1], Ungar [18] and Staněk [17]) proved in the one-dimensional periodic case that $e(\lambda) < 0$ for all $\lambda \neq 0$ if $\int_0^1 V(x)dx = 0$ (note the strict inequality). By a Bloch wave analysis and eigenvalue perturbation theory [13], this result is easy, not only in one dimension but also for ν-dimensional periodic potentials (Eastham [4, 5] only proves $e(\lambda) \leqq 0$) if V is periodic with $\int_{\text{unit cell}} V(x)dx = 0$.