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Self-Similar Constructions in Smooth Dynamics: Rigidity, Smoothness and Dimension

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Abstract. In the first part of this paper, for each $d \geq 2$, we construct diffeomorphisms of the d-dimensional ball which have zero entropy, one periodic orbit with period 2^n for each $n \geq 0$, no other periodic orbits, and a single invariant Cantor set which has a continuum of possible but, in any case, simple geometric structures. These diffeomorphisms are $C^{r(d)}$-smooth, where $r(d)$ is a strictly increasing function of d, which goes to infinity with d. The second part contains a more general result about smooth maps obtained by an infinite sequence of surgeries, and further particular cases.

General Introduction

This paper contains two parts.
– In the first part, we show how a few straightforward ideas combine to give simple smooth maps at the accumulation of cascades of period doubling bifurcations, with a smoothness which gets improved when increasing the dimension (Theorem 1). This has the following consequence:

The minimal smoothness required to hope for universality at the accumulation of period doubling bifurcations increases with the dimension.

– The second part begins by a formulation of a general result, our Theorem 2, which only needs obvious changes to the specific arguments used in the proof of Theorem 1. Then, we give more applications of the general result, dealing with C^2 diffeomorphisms of the two-disk with zero entropy having infinitely many periodic orbits.

Here are three examples in C^k-smooth dynamics with $k < \infty$ which together, serve as a general motivation to the present work. For simplicity, in this general introduction, we only care about C^k when k is an integer, which leaves aside many interesting questions!