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The Equality of Fractal Dimension and Uncertainty Dimension for Certain Dynamical Systems*

Helena E. Nusse[1,2] **and James A. Yorke**[1,3]

[1] Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA
[2] Rijksuniversiteit Groningen, Fac. Economische Wetenschappen, WSN-gebouw, Postbus 800, NL-9700 AV Groningen, The Netherlands
[3] Department of Mathematics, University of Maryland, College Park, MD 20742, USA

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Abstract. [MGOY] introduced the uncertainty dimension as a quantative measure for final state sensitivity in a system. In [MGOY] and [P] it was conjectured that the box-counting dimension equals the uncertainty dimension for basin boundaries in typical dynamical systems. In this paper our main result is that the box-counting dimension, the uncertainty dimension and the Hausdorff dimension are all equal for the basin boundaries of one and two dimensional systems, which are uniformly hyperbolic on their basin boundary. When the box-counting dimension of the basin boundary is large, that is, near the dimension of the phase space, this result implies that even a large decrease in the uncertainty of the position of the initial condition yields only a relatively small decrease in the uncertainty of which basin that initial point is in.

1. Introduction

Nonlinear dynamical systems often have more than one attractor, and it is of fundamental importance to be able to determine which attractor a specified initial condition goes to. We are interested in the basin boundary, that is, common boundary between the basins of the attractors. For example, for suitably chosen parameter values, the Hénon map has a fractal basin boundary between the points whose orbits go to ∞ (infinity is an attractor) and the points whose trajectories remain bounded and go to the chaotic attractor. When the basin boundary is fractal, it follows that there is a non-attracting, compact, chaotic, invariant set in the basin boundary. Examples with fractal basin boundaries are common and occur for example in the forced damped pendulum and the forced Duffing equation.

The fact that a basin boundary is fractal does have important practical consequences. In particular, for the purposes of determining which attractor eventually

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