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Hamiltonian Collapsing of Irrational Lagrangian Submanifolds with Small First Betti Number★

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Abstract. Among the main symplectic invariants of a closed Lagrange submanifold L of the cotangent of $\mathbb{R}^n$ is the tubular radius $R(L)$ defined as the smallest tube $D(r)\times\mathbb{C}^{n-1}$ of $\mathbb{C}^n \simeq T^*\mathbb{R}^n$ in which L can be pushed by an Hamiltonian diffeotopy of $\mathbb{C}^n$. We show here, using pseudo-holomorphic techniques, that such a submanifold cannot collapse if the first Betti number of L is smaller than 3 and if the Maslov class of L does not vanish; in other words, $R(L)$ is then strictly positive and one can actually give an explicit lower bound in terms of the Liouville and Maslov classes of L.

1. Introduction

The study of symplectic invariants of closed Lagrangian submanifolds of cotangent spaces – an essential aspect of the theory of Hamiltonian systems – was confronted by some fundamental difficulties that have been partly understood during the last years. Two different approaches have been used yielding surprisingly similar estimates. The variational approach, for instance, has been used by Floer, Hofer, and Viterbo to obtain explicit values of the "capacity" of some Lagrangian submanifolds and lower bounds on the value of the "tubular capacity" of any Lagrangian embedding of the torus T^n in $\mathbb{C}^n$. Here the *capacity* $c(E)$ of a subset E of $\mathbb{C}^n$ is defined by a minimax procedure (see [2, 3]) and is given, in the simplest case of a closed convex hypersurface of $\mathbb{C}^n$ by the smallest symplectic area of a closed characteristic; the *tubular capacity* $R(E)$ is the smallest radius of a tube into which E can be pushed by an Hamiltonian diffeotopy:

$$R(E) = \inf\{r \in \mathbb{R}^+ \text{ such that there exists an Hamiltonian diffeomorphism } \phi \text{ of } \mathbb{C}^n \text{ with } \phi(E) \subset D(r) \times \mathbb{C}^{n-1}\},$$

where $D(r)$ is the open disk of radius r of $\mathbb{C}$.

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