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Regularization and Convergence for Singular Perturbations

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Abstract. We present an abstract result on removing regularization for singular perturbations in the operator theory. Our main result concerns singular perturbations which are not (formally) semibounded from below.

1. Introduction

For very singular and nonpositive potentials $-V$, $V \geq 0$, in quantum mechanics it may happen that the Schrödinger operator $H = -\Delta - V$ makes no sense, i.e., it is not essentially self-adjoint on $\mathscr{D}(-\Delta) \cap \mathscr{D}(V)$. Further, if we formally define the Schrödinger operator on $\mathscr{D}(-\Delta)$, then H is not semibounded from below. However, one can choose a regularizing sequence $\{V_n\}_{n=1}^{\infty}$ of bounded potentials such that $V_n f \to Vf$ as $n \to \infty$ for certain elements $f \in \mathscr{D}(V)$ and the corresponding sequence $\{H_n\}_{n=1}^{\infty}$ of well-defined Schrödinger operators, i.e. $H_n = -\Delta - V_n$, such that the limit $\text{s-}\lim_{n\to\infty} (H_n - z)^{-1}$, $\text{Im}(z) \neq 0$, exists and defines a self-adjoint operator which can be understood as a regularized Schrödinger operator for the problem $H = -\Delta - V$ [4, 5, 7, 8, 13]. It is interesting to note that the regularizing sequence $\{V_n\}_{n=1}^{\infty}$ itself chooses the "right" regularized Schrödinger operator or, in other words, that the singular perturbation itself forces the "right" operator.

The present paper has the aim to clarify this phenomenon on an abstract operator-theoretical level. Thus, the paper is closely related to [8]. The main result of [8] (Theorem 1) say that if

(i) the regularizing sequence $\{V_n\}_{n=1}^{\infty}$, $\mathscr{D}(V_n) \supset \mathscr{D}(V)$, satisfies $V_n \leq V$,
(ii) there exists a dense subset $\mathscr{D} \subseteq \mathscr{D}(A) \cap \mathscr{D}(V)$ such that

$$\|Vf\| \leq a\|Af\| + b\|f\|, \qquad a < 1, \qquad b < +\infty, \qquad f \in \mathscr{D}, \tag{1}$$

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