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The Pfaffian Line Bundle

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Abstract. We analyze the holomorphic Pfaffian line bundle defined over an infinite dimensional isotropic Grassmannian manifold. Using the infinite dimensional relative Pfaffian, we produce a Fock space structure on the space of holomorphic sections of the dual of this bundle. On this Fock space, an explicit and rigorous construction of the spin representations of the loop groups LO_n is given. We also discuss and prove some facts about the connection between the Pfaffian line bundle over the Grassmannian and the Pfaffian line bundle of a Dirac operator.

1. Introduction

In this paper, we study the Pfaffian line bundle PF over the isotropic Grassmannian manifold of a Hilbert space. This line bundle, which was first defined in [21], is a unique holomorphic square root of the determinant line bundle over the Grassmannian. Here we will use the theory of the infinite dimensional relative Pfaffian developed in [13] and [16] to construct a Hilbert space $\mathscr{F}$ out of the space of holomorphic sections of the dual bundle of PF. If we use the space of square-integrable wave functions on the circle for the underlying Hilbert space, then $\mathscr{F}$ is interpreted as the Fock space of a Majorana fermion on the circle (with half the degrees of freedom of the Dirac Fock space, which arises from the corresponding construction for the determinant line bundle). The physical interpretation of this Pfaffian line bundle Fock space construction was speculated on in [26], and it serves as an example of the Fock space functor described in [24].

The Fock space $\mathscr{F}$ is isomorphic to the completion of an exterior algebra, but the Pfaffian line bundle approach reveals extra structure. The isotropic Grassmannian $\mathrm{Gr}_I(\mathscr{H})$ is a homogeneous space of the restricted orthogonal group $O_{\mathrm{res}}(\mathscr{H})$ associated to a real structure on the Hilbert space $\mathscr{H}$. We show that $\mathscr{F}$ carries a projective unitary representation of $O_{\mathrm{res}}(\mathscr{H})$, which is an analytic generalization of the representations described algebraically in [14]. In analogy to the Borel–Weil

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