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Estimates and Extremals for Zeta Function Determinants on Four-Manifolds*

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Abstract. Let A be a positive integral power of a natural, conformally covariant differential operator on tensor-spinors in a Riemannian manifold. Suppose that A is formally self-adjoint and has positive definite leading symbol. For example, A could be the conformal Laplacian (Yamabe operator) L, or the square of the Dirac operator $\nabla\!\!\!/$. Within the conformal class $\{g = e^{2w}g_0 | w \in C^\infty(M)\}$ of an Einstein, locally symmetric "background" metric g_0 on a compact four-manifold M, we use an exponential Sobolev inequality of Adams to show that bounds on the functional determinant of A and the volume of g imply bounds on the $W^{2,2}$ norm of the conformal factor w, provided that a certain conformally invariant geometric constant $k = k(M, g_0 A)$ is strictly less than $32\pi^2$. We show for the operators L and $\nabla\!\!\!/^2$ that indeed $k < 32\pi^2$ except when (M, g_0) is the standard sphere or a hyperbolic space form. On the sphere, a centering argument allows us to obtain a bound of the same type, despite the fact that k is exactly equal to $32\pi^2$ in this case. Finally, we use an inequality of Beckner to show that in the conformal class of the standard four-sphere, the determinant of L or of $\nabla\!\!\!/^2$ is extremized exactly at the standard metric and its images under the conformal transformation group $O(5, 1)$.

1. Introduction and Statement of Results

On a compact Riemannian manifold (M, g), there are many *natural*, or *geometric* elliptic operators associated with the metric; for example the Laplacian $\Delta = -g^{-1/2}\partial_i(g^{ij}g^{1/2}\partial_j)$ on functions. When we wish to emphasize the underlying conformal structure, it is natural to consider operators which transform in a simple manner under conformal change of metric. A *conformally covariant* operator is a geometric differential operator A which undergoes the following transformation

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