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On Mod 2 and Higher Elliptic Genera

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1. Abstract. In the first part of this paper, we construct mod 2 elliptic genera on manifolds of dimensions $8k + 1$, $8k + 2$ by mod 2 index formulas of Dirac operators. They are given by mod 2 modular forms or mod 2 automorphic functions. We also obtain an integral formula for the mod 2 index of the Dirac operator. As a by-product we find topological obstructions to group actions. In the second part, we construct higher elliptic genera and prove some of their rigidity properties under group actions. In the third part we write down characteristic series for all Witten genera by Jacobi theta-functions. The modular property and transformation formulas of elliptic genera then follow easily. We shall also prove that Krichever's genera, which come from integrable systems, can be written as indices of twisted Dirac operators for SU-manifolds. Some general discussions about elliptic genera are given.

2. Introduction

Elliptic genera were first constructed by Ochanine [28], Landweber-Stong in a topological way. Witten gave a geometric interpretation of them. More precisely, he showed that the Lefschetz fixed point formula of twisted Dirac operator on loop space gave the universal elliptic genera. Recently Krichever [20] derived certain elliptic genera from the theory of KP equations. Their constructions work for manifolds of dimension $8k$ and $8k + 4$ or almost complex manifolds. Later Ochanine constructed mod 2 elliptic genera for manifolds of dimension $8k + 1$ and $8k + 2$ by cobordism theory in [29]. This construction was conjectured by Landweber [21].

In this paper, we construct mod 2 elliptic genera by the Atiyah-Singer mod 2 index formula. We take Witten's point of view to give a geometric construction of mod 2 elliptic genera. The method is quite easy. We lift the classical index formula to a KO-invariant version, and check that it is equal to the KO-invariant in cobordism theory. Hence we can borrow some beautiful ideas from cobordism theory. Theorem 1 and Corollary 1 answer a question of Witten in [36], viz. the