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Zeta Function Continuation and the Casimir Energy on Odd and Even Dimensional Spheres

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Abstract. The zeta function continuation method is applied to compute the Casimir energy on spheres S^N. Both odd and even dimensional spheres are studied. For the appropriate conformally modified Laplacian Δ the Casimir energy $\mathscr{E}$ is shown to be finite for all dimensions even though, generically, it should diverge in odd dimensions due to the presence of a pole in the associated zeta function $\zeta_\Delta(s)$. The residue of this pole is computed and shown to vanish in our case. An explicit analytic continuation of $\zeta_\Delta(s)$ is constructed for all values of N. This enables us to calculate $\mathscr{E}$ exactly and we find that the Casimir energy vanishes in all even dimensions. For odd dimensions $\mathscr{E}$ is never zero but alternates in sign as N increases through odd values. Some results are also derived for the Casimir energy of other operators of Laplacian type.

1. Introduction

The treatment of relativistic quantum fields in curved spaces is a notoriously difficult problem which has only yielded partial results to date. In particular the evaluation of the energy momentum tensor for free fields has an involved and somewhat chequered history (for a review see Birrell and Davies [1]). A particularly useful tool in this regard is the heat equation for an elliptic differential operator P and the corresponding zeta function $\zeta_P(s)$ (Gilkey [2]). The usefulness of the heat equation for the determination of effective action in quantum field theory was emphasised in Hawking [3]. An explicit evaluation of the zeta function is only possible on spaces with a high degree of symmetry and it is worthwhile examining such situations in detail in the hope of obtaining insights into its more general properties. For example the zeta function regularisation method was used by Dowker and Banach [4] for an evaluation of the Casimir energy on S^3.

In this paper the zeta function will be explicitly evaluated at the particular value of its argument $s = -1/2$ and some of its properties will be examined for a class of elliptic operators on N-dimensional spheres. The value $s = -1/2$ is important because it yields the Casimir energy of S^N. We shall use the $SO(N+1)$ symmetric metric on the sphere with radius a, and the elliptic operator considered will be the