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Determinants, Finite-Difference Operators and Boundary Value Problems*

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Abstract. We relate the determinants of differential and difference operators to the boundary values of solutions of the operators. Previous proofs of related results have involved considering one-parameter families of such operators, showing the desired quantities are equal up to a constant, and then calculating the constant. We take a more direct approach. For a fixed operator, we prove immediately that the two sides of our formulas are equal by using the following simple observation (Proposition 1.3): *Let $U \in SU(n, \mathbf{C})$. Write U in block form*

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix},$$

where u_{11} and u_{22} are square matrices. Then

$$\det u_{11} = \overline{\det u_{22}} .$$

0. Introduction

Motivated by questions in quantum field theory, there has been much recent interest in the problem of calculating the determinant of differential operators (see, for example, [Ra] chapter III). Suppose L is a positive elliptic differential operator acting on sections of a vector bundle over a compact manifold. Then L has a discrete spectrum

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \to \infty .$$

Various methods have been used to make sense of

$$\det L \text{ “} = \Pi \lambda_i \text{”} .$$

Perhaps the most common method is the zeta-function regularization of Ray and Singer [R-S], in which one defines $\log \det L$ by analytically continuing the function

$$\sum \lambda_i^{-s} \log \lambda_i$$

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