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Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function*

Maxim Kontsevich

Max-Planck Institut für Mathematik, Gottfried-Claren-Straße 26, W-5300 Bonn 1, Federal Republic of Germany

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Abstract. We show that two natural approaches to quantum gravity coincide. This identity is nontrivial and relies on the equivalence of each approach to KdV equations. We also investigate related mathematical problems.

1. Witten's Conjecture

1.1. Two-Dimensional Gravity(ies). Quantum gravity, although not well-defined, looks like integration over the (infinite-dimensional) space of riemannian metrics on manifolds modulo diffeomorphisms. There are at least two mathematically consistent approaches to two-dimensional gravity.

The first one was developed by [KB, DS, GM] and can be called "enumeration of triangulations." Any triangulation of the surface determines some singular metric obtained from the arrangement of equilateral triangles. One can imagine that when the number of triangles tends to infinity these singular metrics approximate "random metrics" on surfaces. Thus we are led to the problem of finding the asymptotics of the number of triangulations of surfaces of fixed genus into the given growing number of triangles. It was shown (using Feynman diagram techniques) that this problem together with some modifications is equivalent to describing the asymptotic behaviour of the integrals $\int \exp(\mathrm{tr}\, P(X))\, dX$, where X runs over the space of hermitian $N \times N$-matrices, $N \to \infty$ and P is a polynomial depending (in some way) on N. These integrals were evaluated using orthogonal polynomials. It turns out that discrete Toda lattice equations hold. In the limit the Korteweg-de Vries equation arises. The partition function of the two-dimensional gravity for this approach is a series in an infinite number of variables and coincides with the logarithm of some τ-function for KdV-hierarchy.

Another approach is to choose some specific action. Using supersymmetry the integral over the space of all metrics reduces to the integral over the finite-dimensional space of conformal structures. The last integral has a cohomological description as an intersection theory on the compactified moduli space of

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