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Operator Equalities Related to the Quantum $E(2)$ Group*

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Abstract. The paper deals with normal operators R and S satisfying simple commutation relations that we encounter investigating the quantum deformation of the $E(2)$ group. We show that $R+S$ admits a normal extension if and only if $R^{-1}S$ satisfies a certain spectral condition. A number of related formulae are derived. In particular, all the functions f satisfying the character equation $f(R+S)=f(R)f(S)$ are found.

0. Introduction

Investigating the quantum deformation (of the two-fold covering) of the group of motions of the Euclidean plane $E(2)$ we often deal with pairs of normal operators (R, S) satisfying in a strong sense the relations $SR=\mu^2 RS$ and $SR^*=R^*S$ (where μ is a real number, $0<\mu<1$). In the present paper we prove a number of results involving such operators. The results will be used in [3], where the quantum $E(2)$ and its Pontryagin dual are elaborated and in [4], where a quantum deformation of the Lorentz group is introduced and investigated.

Let μ be a real number such that $0<\mu<1$. We denote by D_μ the set of all pairs (R, S) of normal operators acting on a Hilbert space H satisfying the following five conditions:

$$\left.\begin{aligned}
&1.\ \operatorname{Ker}R=\operatorname{Ker}S=\{0\},\\
&2.\ (\text{Phase } R)(\text{Phase } S)=(\text{Phase } S)(\text{Phase } R),\\
&3.\ (\text{Phase } R)^*|S|(\text{Phase } R)=\mu|S|,\\
&4.\ (\text{Phase } S)|R|(\text{Phase } S)^*=\mu|R|,\\
&5.\ |S| \text{ and } |R| \text{ strongly commute}.
\end{aligned}\right\} \tag{0.1}$$

One can easily show the following

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