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Complements to Various Stone-Weierstrass Theorems for C^*-algebras and a Theorem of Shultz

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Abstract. J. Glimm's Stone-Weierstrass theorem states that if A is a C^*-algebra, $P(A)$ is the set of pure states of A, and B is a C^*-subalgebra which separates $\overline{P(A)} \cup \{0\}$, then $B = A$. We show that if B is a C^*-subalgebra of A and x an element of A such that any two elements of $\overline{P(A)} \cup \{0\}$ which agree on B agree also on x, then $x \in B$. Similar complements are given to other Stone-Weierstrass theorems. A theorem of F. Shultz states that if $x \in A^{**}$, the enveloping von Neumann algebra of A, and if x, x^*x, and xx^* are uniformly continuous on $P(A) \cup \{0\}$, then there is an element of A which agrees with x on $P(A)$. We show that the hypotheses on x^*x and xx^* can be dropped.

The Stone-Weierstrass conjecture is that if B is a C^*-subalgebra of A and if B separates $P(A) \cup \{0\}$, then $B = A$. It was shown essentially by Kaplansky (see p. 16 of [2] for the history of this result) that this is true if A or B is GCR. It was shown by Sakai [18] that the conjecture is true if B is separable and nuclear. In Theorem 5(a) below we show that if $x \in A$, if any two elements of $P(A) \cup \{0\}$ which agree on B agree also on x, and if one of the above extra hypotheses is satisfied, then $x \in B$. Theorem 5(a) also implies single-element versions of some other Stone-Weierstrass theorems that have been proved, and possibly others that will be proved.

The factorial Stone-Weierstrass conjecture is that if B is a C^*-subalgebra of A which separates $F(A) \cup \{0\}$, where $F(A)$ is the set of factorial states of A, then $B = A$. This was proved in the separable case by Longo [14] and Popa [16], independently. In Theorem 5(b) below we show that if B is separable and if any two elements of $F(A) \cup \{0\}$ which agree on B agree also on x, then $x \in B$. A major part of the proofs of the factorial conjecture was the solution of the factorial state extension problem. Theorem 6.1 of [14] states that if B is separable, then any factorial state of B extends to a factorial state of A. (Theorem 4 of [16] states the same result for A separable.) This result as well as the factorial Stone-Weierstrass theorem itself is used in our proof.

Glimm's Stone-Weierstrass theorem appears in [12], and our complement to it, stated in the abstract, is Theorem 5(c) below.