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Differential Equations in the Spectral Parameter, Darboux Transformations and a Hierarchy of Master Symmetries for KdV

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Abstract. We study a certain family of Schrödinger operators whose eigenfunctions $\varphi(x,\lambda)$ satisfy a differential equation in the spectral parameter λ of the form $B(\lambda,\partial_\lambda)\varphi=\Theta(x)\varphi$. We show that the flows of a hierarchy of master symmetries for KdV are tangent to the manifolds that compose the strata of this class of *bispectral* potentials. This extends and complements a result of Duistermaat and Grünbaum concerning a similar property for the Adler and Moser potentials and the flows of the KdV hierarchy.

1. Introduction

The theory of solitons is still a source of surprises and unexpected connections. The purpose of this article is to report another one of these connections. More specifically, the link between a hierarchy of nonlinear evolution equations closely related to the Korteweg-de Vries (KdV) equation and the *bispectral problem*, which was introduced in [8]. This problem, for the Schrödinger operator $L=-\partial_x^2+u$, can be formulated as follows: When do the solutions $\varphi(x,\lambda)$ of

$$L\varphi=\lambda\varphi \tag{1}$$

also satisfy a differential equation in the spectral parameter λ of the form

$$B(\lambda,\partial_\lambda)\varphi=\Theta(x)\varphi\,, \tag{2}$$

where $B(\lambda,\partial_\lambda)$ is a differential operator of positive order and $\Theta(x)$ is independent of λ? The solution to this problem, under very mild assumptions on $u(x)$, turns out to be related to the theory of the KdV equation

$$u_t=-u_{xxx}+6u_xu \tag{3}$$

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