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Using Integrability to Produce Chaos: Billiards with Positive Entropy

Victor J. Donnay*

Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010, USA

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Abstract. A new open class of convex 2 dimensional planar billiards with positive Lyapunov exponent almost everywhere is constructed. We introduce the notion of a focusing arc and show that such arcs can be used to build billiard systems with positive Lyapunov exponents. We prove that under small C^6 perturbations, focusing arcs remain focusing and thereby show that perturbations of the Bunimovich stadium billiard have positive Lyapunov exponents.

0. Introduction

We study the ergodic properties of billiards inside a planar domain Q for which the boundary ∂Q consists of piecewise smooth arcs that are either flat or convex. A billiard in Q is the dynamical system arising from the uniform motion of a point mass inside Q with elastic reflections at the boundary: angle of reflection equals angle of incidence. We introduce a very general class of convex arcs for which the resulting billiard will have positive Lyapunov exponents almost everywhere and hence by Pesin [P] will have positive measure-theoretic entropy (chaos). This class is open in the C^6 topology on curves. Using these results, we prove that C^6 small perturbations of the Bunimovich stadium billiard have positive Lyapunov exponents. Our examples generalize work of Bunimovich [B3], Wojtkowski [W3], and Markarian [Mr1, Mr2] who had described various non-open classes of arcs.

We say a non-closed curve is convex if when we connect the ends of the curve by a straight line, the resulting closed curve has no self-intersections and bounds a convex set. Henceforth, the term convex curve will signify a C^∞ smooth curve that is non-closed and whose curvature is never zero, i.e. the curve is strictly convex.

Let γ be a convex curve. We examine a ray that collides with γ a finite number of times.

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