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Persistently Expansive Geodesic Flows

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Abstract. We prove that C^1-persistently expansive geodesic flows of compact, boundaryless Riemannian manifolds have the property that the closure of the set of closed orbits is a hyperbolic set. In the case of compact surfaces we deduce that the geodesic flow is C^1-persistently expansive if and only if it is an Anosov flow.

Introduction

In this paper we present some results concerning geodesic flows possessing certain topological properties which persist under small perturbations. Recall that if (M, g) is a complete Riemannian manifold and T_1M is its unit tangent bundle, the geodesic flow $\varphi_t\colon T_1M \to T_1M$ is defined as follows: given a point $(p, v) \in T_1M$, $\varphi_t(p, v) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is the unit geodesic of M such that $\gamma(0) = p$ and $\gamma'(0) = v$. Let us denote as $\kappa^k(M)$ the set of geodesic flows of Riemannian metrics of M endowed with the C^k topology. Given any one parameter family of homeomorphisms $\psi_t\colon N \to N$ acting on a metric space N, we say that it is *expansive* if there is an $\varepsilon > 0$ such that every $p \in N$ satisfies the following property: if $q \in N$ and there exists a continuous surjection $f_q\colon \mathbb{R} \to \mathbb{R}$ with $d(\psi_t(p), \psi_{f_q(t)}(q)) \leqq \varepsilon$ for every $t \in \mathbb{R}$, then there exists $t_0 \in \mathbb{R}$ depending on ε, p, q, with $t_0 \to 0$ if $d(p, q) \to 0$ such that $q = \psi_{t_0}(p)$. When $t \in \mathbb{Z}$ for every t we just take $f_q(t) = t$ and $t_0 = 0$.

The persistence of expansivity is closely related with hyperbolicity and stability of dynamical systems. Let $E^k(M)$ be the subset of $\kappa^k(M)$ of expansive geodesic flows. An Anosov geodesic flow of a compact manifold M is expansive, and since it is C^1-structurally stable [1] it belongs to $\operatorname{int}(E^1(M))$ – the interior of $E^1(M)$ in $\kappa^1(M)$. Axiom A systems are expansive near the closure of the set of periodic orbits, and since they are Ω-stable [10, 12] this property persists under C^1 perturbations. On the other hand, Mañe [6] proves that the interior of the set of expansive diffeomorphisms in $\operatorname{Diff}^1(M)$ (i.e. the set of C^∞ diffeomorphisms of M endowed with the C^1 topology) coincides with the