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Exponentially Small Adiabatic Invariant for the Schrödinger Equation*

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Abstract. We study an adiabatic invariant for the time-dependent Schrödinger equation which gives the transition probability across a gap from time t' to time t. When the hamiltonian depends analytically on time, and $t' = -\infty$, $t = +\infty$ we give sufficient conditions so that this adiabatic invariant tends to zero exponentially fast in the adiabatic limit.

1. Introduction

Let $H(t), t \in \mathbb{R}$, be a self-adjoint operator on a Hilbert space $\mathscr{H}$. We study the time-dependent Schrödinger equation in the adiabatic limit, i.e.

$$i\varepsilon \frac{\partial}{\partial t} \varphi(t) = H(t)\varphi(t), \quad t \in \mathbb{R} \tag{1.1}$$

when $\varepsilon \to 0$. The self-adjoint operator $H(t)$ satisfies three conditions.

I. Self-Adjointness and Analyticity. There exists a band S_a in the complex plane, $S_a = \{t + is : |s| < a\}$, and a dense domain $D \subset \mathscr{H}$ such that for each $z \in S_a, H(z)$ is a closed operator defined on $D, H(z)\varphi$ is holomorphic on S_a for each $\varphi \in D$ and $H(z)^* = H(\bar{z})$. Moreover we suppose that $H(t)$ is bounded from below for $t \in \mathbb{R}$.

II. Behaviour at Infinity. There exist two self-adjoint operators H^+ and H^-, bounded from below and defined on D, two positive constants C and α such that for all $\varphi \in D$ and $|t|$ large enough

$$\sup_{|s|<a} \|(H(t+is) - H^+)\varphi\| \leqq \frac{C}{(1+|t|)^{1+\alpha}} (\|\varphi\| + \|H^+\varphi\|), \quad t > 0$$

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