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A Local Index Theorem for Families of $\bar{\partial}$-Operators on Punctured Riemann Surfaces and a New Kähler Metric on Their Moduli Spaces

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Abstract. We prove a local index theorem for families of $\bar{\partial}$-operators on Riemann surfaces of type (g, n), i.e. of genus g with $n > 0$ punctures. We calculate the first Chern form of the determinant line bundle on the Teichmüller space $T_{g,n}$ endowed with Quillen's metric (where the role of the determinant of the Laplace operators is played by the values of the Selberg zeta function at integer points). The result differs from the case of compact Riemann surfaces by an additional term, which turns out to be the Kähler form of a new Kähler metric on the moduli space of punctured Riemann surfaces. As a corollary of this result we derive, for instance, an analog of Mumford's isomorphism in the case of the universal curve.

Introduction

The Atiyah-Singer index theorem for families of elliptic operators, which plays an important role in modern mathematical physics, is of particular interest for $\bar{\partial}$-operators on complex manifolds. Consider a holomorphic family $p : \mathscr{X} \to B$ of compact complex manifolds over a compact base B, and a holomorphic vector bundle $\mathscr{E} \to \mathscr{X}$. The family $\bar{\partial} = \{\bar{\partial}_b\}_{b\in B}$ of $\bar{\partial}$-operators in the vector bundles $E_b \to X_b$ (restrictions of $\mathscr{E}$ over the fibers $X_b = p^{-1}(b)$, $b \in B$) gives rise (in the sense of K-theory) to the index bundle $\operatorname{ind}\bar{\partial} \in K(B)$ on B with fibers $\ker\bar{\partial}_b - \operatorname{coker}\bar{\partial}_b$ over $b \in B$. The Atiyah-Singer index theorem applied to this special case states that

$$\operatorname{ch}(\operatorname{ind}\bar{\partial}) = p_*(\operatorname{ch}\mathscr{E} \cdot \operatorname{td} T_v\mathscr{X}) . \tag{1}$$

Here ch denotes the Chern character, $\operatorname{td} T_v\mathscr{X}$ is the Todd class of the vertical tangent (along the fibers of $p : \mathscr{X} \to B$) bundle on $\mathscr{X}$, and $p_* : H^*(\mathscr{X}) \to H^{*-\dim X_b}(B)$ is the operation of "integration along the fibers" (see [1]).

In many applications the bundles $\mathscr{E}$ and $T_v\mathscr{X}$ are Hermitian, so that each of them carries the (canonical) unitary connection compatible with the holomorphic