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Rank of Quantized Universal Enveloping Algebras and Modular Functions

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Abstract. We compute an intrinsic rank invariant for quasitriangular Hopf algebras in the case of general quantum groups $U_q(g)$. As a function of q the rank has remarkable number theoretic properties connected with modular covariance and Galois theory. A number of examples are treated in detail, including rank $(U_q(su(3)))$ and rank $(U_q(e_8))$. We briefly indicate a physical interpretation as relating Chern–Simons theory with the theory of a quantum particle confined to an alcove of g.

1. Introduction

Quasitriangular Hopf algebras (or "quantum groups") are a close generalization of groups. In particular, for finite-dimensional quasitriangular Hopf algebras H there is a natural invariant, rank(H), arising out of category theory as a natural generalization of the order, $|G|$, of a finite group G [19, 18]. The definition is recalled in the preliminaries below. Moreover, the rank extends in a natural way to the infinite-dimensional quantum groups $U_q(g)$. It was computed explicitly for $g = su(2)$ as [19],

$$\operatorname{rank}(U_q(su(2))) = \frac{1 + 2\sum_{n\in\mathbb{N}} q^{-(1/2)n^2}}{1 - q^{-2}}. \tag{1}$$

The numerator here is a theta function $\theta(q^{-1/2})$ and has remarkable modular transformation properties under $\tau \to -\frac{1}{\tau}$. Here $q = e^{2\pi i\tau}$ and explicitly, $\theta(e^{-\pi i/\tau}) = (i\tau)^{1/2}\theta(e^{\pi i\tau})$. Because rank is an invariant, this suggests the possibility of simple transformation properties of the algebra $U_q(su(2))$ as q undergoes a modular

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