Commun. Math. Phys. 136, 599–605 (1991)

Communications in Mathematical Physics

On Ergodic One-Dimensional Cellular Automata*

M. Shirvani and T. D. Rogers

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada

Received May 10, 1989

Abstract. We show that all onto cellular automata defined on the binary sequence space are invariant with respect to the Haar measure, and that an extensive class of such maps (including many nonlinear ones) are strongly mixing with respect to the Haar measure.

I. Introduction

Let X denote the space of bi-infinite sequences $a=(a_i)_{i\in\mathbb{Z}}$, where each $a_i=0$ or 1, regarded as a compact abelian group under component-wise addition. Denote the normalized Haar measure on X by μ. Let σ be the shift map defined by $\sigma(a)_i=a_{i+1}$ for all $i\in\mathbb{Z}$ and all $a\in X$. If $f:\{0,1\}^n\to\{0,1\}$ is a Boolean function of n variables and $r\leqq s$ are fixed integers with $s-r=n+1$, then we write f_∞ for the corresponding cellular automaton: $f_\infty(a)_i=f(a_{i+r},\ldots,a_{i+s})$ for all $i\in\mathbb{Z}$. Surjective such maps have been analyzed in great detail from both the combinatorial and the topological points of view [1, 3, 8]. We characterize those f_∞ which preserve the Haar measure [i.e. $\mu(f_\infty^{-1}(A))=\mu(A)$ for all measurable subsets A of X] in Theorem 2.4, in particular showing that f_∞ is onto if and only if it preserves the Haar measure. (The latter result was announced by J. Milnor in [2].) We show further that certain of the f_∞ are actually ergodic with respect to μ (Theorems 3.2 and 3.4), although our results here are not complete since we suspect that all onto one-dimensional cellular automata (with the exception of the identity and the inversion map) are ergodic with respect to μ. Nonetheless, 3.4 shows that certain nonlinear automata considered by Wolfram in [7, Chap. 2.3], are in fact strongly mixing. To our knowledge these are the first examples of nonlinear ergodic automata in the literature.

* This work was supported in part by grants from NSERC of Canada