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Proof of the Landau–Zener Formula in an Adiabatic Limit with Small Eigenvalue Gaps

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Abstract. We consider a smooth operator-valued function $H(t,\delta)$ that has two isolated non-degenerate eigenvalues $E_{\mathscr{A}}(t,\delta)$ and $E_{\mathscr{B}}(t,\delta)$ for $\delta > 0$. We assume these eigenvalues are bounded away from the rest of the spectrum of $H(t,\delta)$, but have an avoided crossing with one another with a closest approach that is $O(\delta)$ as δ tends to zero. Under these circumstances, we study the small ε limit for the adiabatic Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = H(t,\varepsilon^{1/2})\psi.$$

We prove that the Landau–Zener formula correctly describes the coupling between the adiabatic states associated with the eigenvalues $E_{\mathscr{A}}(t,\delta)$ and $E_{\mathscr{B}}(t,\delta)$ as the system propagates through the avoided crossing.

1. Introduction

Adiabatic approximations in quantum mechanics describe solutions to Schrödinger equations with slowly varying time-dependent Hamiltonians. More precisely, if the time scale is chosen to be commensurate with the Hamiltonian's variation, then adiabatic approximations describe the small ε behavior of solutions to the Schrödinger equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = H(t)\psi, \tag{1.1}$$

for t in some fixed interval. The classical adiabatic theorem states that if $H(t)$ is a smooth family of self-adjoint operators with a continuous, isolated, multiplicity

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