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Unbounded Elements Affiliated with C^*-Algebras and Non-Compact Quantum Groups**

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Abstract. The affiliation relation that allows to include unbounded elements (operators) into the C^*-algebra framework is introduced, investigated and applied to the quantum group theory. The quantum deformation of (the two-fold covering of) the group of motions of Euclidean plane is constructed. A remarkable radius quantization is discovered. It is also shown that the quantum $SU(1,1)$ group does not exist on the C^*-algebra level for real value of the deformation parameter.

0. Introduction

In practical computations in quantum physics we mostly deal with unbounded physical quantities represented by unbounded operators. On the other hand in the very theoretical approaches (see for example [5, 2]) we consider C^*-algebras consisting of bounded elements only. Therefore it is necessary to investigate the relation between particular unbounded operators and C^*-algebras.

The same problem in a more apparent way arises in the theory of non-compact topological quantum groups, where on the one hand the doctrine [18] says that the C^*-algebra language is the only one to be used and where on the other hand we have to deal with matrix elements of finite-dimensional non-unitary representations which in general are not bounded.

The similar problem was encountered in the von Neumann algebra theory [11] where the affiliation relation $a\eta M$ [where $M \subset B(H)$ is a von Neumann algebra and a is an unbounded operator acting on the Hilbert space H] was invented to describe such situations. We borrow from this theory the name of the relation and its symbol: in what follows we shall speak about unbounded elements a affiliated with a C^*-algebra A and write $a\eta A$. We have however to warn the reader that the affiliation relation that we introduce in the present paper is not a generalization of

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