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Metrics on the Moduli Spaces of Instantons Over Euclidean 4-Space

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Abstract. We prove that the natural hyper-Kähler metrics on the moduli space of charge k instantons over Euclidean four-space and on the space of ADHM matrices coincide. We use this to deduce formulae relating expressions in the curvature of a connection to invariant polynomials in the ADHM matrices corresponding to this connection. These arise from consideration of the group of symmetries acting on the moduli spaces.

Introduction

The ADHM construction of instantons identifies the moduli space of charge k $SU(r)$ instantons over the 4-sphere with a space of complex matrices arising from monads. If we consider framed instantons over $\mathbb{R}^4$ instead, these spaces $\mathcal{M}_{k,r}$ and $\hat{\mathcal{M}}_{r,k}$ respectively, have dimension $4rk$ and are well known to admit hyper-Kähler metrics, and it has been supposed, as a sort of folk result, that the ADHM correspondence is actually a hyper-Kähler isometry. It is one of the aims of this paper to prove this result.

Our proof boils down to identifying suitable hyper-Kähler potentials for the metrics on $\mathcal{M}_{k,r}$ and $\hat{\mathcal{M}}_{r,k}$ and proving the apparently stronger result that these potentials agree. In fact, we can view the hyper-Kähler potential on $\mathcal{M}_{k,r}$ as a potential for the dilation action of $\mathbb{R}^4$ lifted to the moduli space. This links our isometry with the work of Groisser and Parker [9, 10]. As an extension of this we shall prove a general result which describes potential functions for subgroups of the conformal group acting on $\mathcal{M}_{k,r}$ in terms of the potential functions for the same groups acting on $\mathbb{R}^4$. We observe that the hyper-Kähler potential is also the moment map for a circle action on $\mathcal{M}_{k,r}$ (also lifted from $\mathbb{R}^4$) and deduce a formula for other moment maps of other groups of isometries of $\mathcal{M}_{k,r}$. Since we have

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