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Preservation of Logarithmic Concavity by the Mellin Transform and Applications to the Schrödinger Equation for Certain Classes of Potentials

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Abstract. We prove that the Mellin transform of a function log-concave (convex) is, after division by $\Gamma(\nu+1)$, where ν is the argument of the transform, itself log-concave (convex) in ν. This theorem is first applied to the moments of the ground state wave function of the Schrödinger equation where the Laplacian of the central potential has a given sign, and generalized to other situations. This is used to derive inequalities linking the ℓ^{th} derivative of the ground state wave function at the origin for angular momentum ℓ and the expectation value of the kinetic energy, and applied to quarkonium physics. A generalization to higher radial excitations is shown to be plausible by using the WKB approximation. Finally, new bounds on ground-state energies in power potentials are obtained.

1. Physical Motivation

In 1984, Baumgartner, Grosse and Martin [1] proved that if, in the Schrödinger equation, the central potential has a Laplacian with a given sign, the order of levels corresponding to what would be a degenerate multiplet for the Coulomb potential is known. Specifically the multiplet is characterized by $N=\ell+n+1=\text{const}$, ℓ being the angular momentum and n the number of nodes of the radial wave function. Then if $\Delta V>0$, the energies decrease when ℓ increases for fixed N, and if $\Delta V<0$ the energies increase when ℓ increases.

A crucial lemma to prove this theorem is the following:

If $\Delta V \gtrless 0$, $\forall r>0$,

$$-\left(\frac{u_\ell'}{u_\ell}\right)' - \frac{\ell+1}{r^2} \gtrless 0, \quad \forall r>0, \tag{1}$$