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Flat Twistor Spaces, Conformally Flat Manifolds and Four-Dimensional Field Theory

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Abstract. A definition is proposed of "four-dimensional conformal field theory" in which the Riemann surfaces of two-dimensional CFT are replaced by (Riemannian) conformally flat four-manifolds and the holomorphic functions are replaced by solutions of the Dirac equation. The definition is investigated from the point of view of twistor theory, allowing methods from complex analysis to be employed. The paper fills in the main mathematical details omitted from the preliminary announcement [15].

1. Introduction

In [23], Segal gives axioms for the notion of two-dimensional conformal field theory (CFT): the basic components are the (conformal equivalence classes of) compact Riemann surfaces Σ with parametrized boundary, together with the natural operations of disjoint union

$$(\Sigma_1, \Sigma_2) \mapsto \Sigma_1 \coprod \Sigma_2, \tag{1.1}$$

and contraction

$$\Sigma \mapsto \Sigma', \tag{1.2}$$

where Σ' is obtained from Σ by using the parametrization to attach a pair of boundary circles to each other.

A two-dimensional conformal field theory is then defined as a "Hilbert-space representation" of (1.1) and (1.2), that is a functor ρ with the following properties: there is a Hilbert space $\mathscr{H}$ with

$$\rho(S^1) = \mathscr{H}; \tag{1.3}$$

and if Σ has p positively oriented and q negatively oriented boundary circles then

$$\rho(\Sigma): \mathscr{H}^{\otimes p} \otimes \bar{\mathscr{H}}^{\otimes q} \to \mathbb{C} \tag{1.4}$$

is a trace-class multilinear functional which satisfies

$$\rho(\Sigma_1 \coprod \Sigma_2) = \rho(\Sigma_1) \otimes \rho(\Sigma_2) \tag{1.5}$$